20.17

THE

MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc., AND PROF. H. W. LLOYD-TANNER, M.A., F.R.S.

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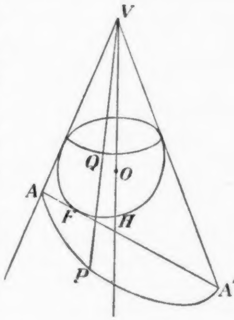
L^2. 2. g.; M^3. 6. e.]

NOTE ON THE SPHERO-CONIC.

Definitions. (i.) A *quadric cone* is a cone having any conic for base, and a point O on the line perpendicular to the plane of the conic through the focus for vertex.

(ii.) A *sphero-conic* is the curve in which any quadric cone cuts a sphere whose centre is the vertex of the cone.

In an introductory course on space-geometry, where only a passing mention of sphero-conics seems possible, I have found it useful to show directly the fundamental property that the sum or difference of the focal distances is constant, instead of following the classical method, which consists in deducing the property by the use of infinitesimals, from the fact that these focal distances make equal angles with the curve; this latter fact being in effect a direct consequence of Desargue's theorem.



The direct demonstration is as follows. We take the well-known figure of a sphere whose centre is O inscribed in a right cone whose vertex is U, and touched by a plane at F. H is a point where the axis of the cone UO meets the sphere, and any generator of the cone meets the circle of contact of the sphere and cone at Q and the conic at P.

Since PF and PQ are tangents of the sphere, $\angle FOP = \angle QOP$.

Therefore $\angle FOP + \angle POH = \angle QOP + \angle POH =$ sum of two angles in the same plane $= \angle QOH$,

$=$ a constant.

Therefore the curve cut from the sphere by the cone whose vertex is O and base the conic is such that the sum of the distances on the sphere from any point to the fixed points F and H is constant.

Such a proof would seem to be in place in books on Geometrical Conics, in which case one might add that the lines OF and OH are called focal lines of the quadric cone.

[The above definition of a quadric cone is evidently equivalent to the following:

The locus of a point whose distance from a fixed line (the *focal axis*) is proportional to its distance from a fixed plane (the *directrix plane*) is called a quadric cone. In the figure the focal axis is the line OF, and the directrix plane is the plane through O and the directrix of the base conic APA'.

From its analogy with the usual definition of a conic, this would seem to be the best form for a metrical definition of a quadric cone. While there is no use in developing to any great extent a synthetic theory of the cone, yet it might be well to say something about it in synthetic books on Conics. For a projective account reference should be made to the translation by Holgate of Reye's "Geometrie der Lage" (Macmillan).]

FRANK MORLEY.

[R. 1. a.]

ON THE EXPRESSION, "MOTION AT AN INSTANT."

Abstract of a Paper read before the Association, Jan. 21, 1899.

IN a recent review[1] of Prof. W. B. Smith's "Infinitesimal Calculus," fault is found with his statement that "In all strictness there can be no *motion at an instant*, and hence no *speed* (or velocity) *at an instant*. The concept of speed (or velocity) or motion will not combine with the concept of instant (or point of time) to form a compound concept." Part of the reviewer's criticism is as follows: "If we allow that motion at an instant is impossible, how are we to escape from Zeno's paradoxical conclusion that all motion is impossible? How can I move from one place to another during a minute, say, if *at every instant* of that interval motion is impossible?"

Now, it cannot be too clearly insisted that an instant does not denote an infinitesimally short period, but that it has *no* duration. It is not to be thought of as in any way resembling the limit of an interval which decreases without limit. Whilst by motion we denote, as defined by Clerk Maxwell, "a change of configuration taking place during a certain time, and in a con-

[1] *Nature*, Dec. 15, 1898.

tinuous manner." It would seem then that the concept *motion* demands extension in time, whilst the concept *instant* denies it, and so I find, with Prof. Smith, that the two will not combine.

The paradox to which the reviewer refers is stated thus[1]: "So long as anything is in one and the same space it is at rest. Hence an arrow is at rest during every moment of its flight, and therefore also during the whole of its flight." The difficulty here lies in the sense to be attributed to the word moment. "During" implies duration, and hence it should denote an interval, but the arrow does not occupy the same position during any interval, however short. At an instant it occupies only one position; no point on it is in two places at once; but at any other instant, however near, it occupies a different position, and the conclusion that the arrow is at rest "during the whole of its flight" would only follow if by any repetition of the instants at each of which it is in one position, we could obtain the interval during which the flight lasts. This we cannot do. Numerical multiplication, whether infinite or not, is insufficient to convert an instant into an interval. The same fallacy of confusion would seem to underlie the reviewer's question quoted above. Either the minute is made up of instants, which would then be infinitesimal intervals, during each of which motion is possible; or else the two clauses of the sentence are mutually irrelevant.

There is a sense in which we do talk of "motion at an instant," and a definition of the expression is given in all text-books of mechanics. The following is due to Clifford: "If there is a certain velocity to which the mean velocity during the interval succeeding a given instant can be made to approach as near as we like by taking the interval small enough, then that velocity is called the instantaneous velocity of the body at the given instant." In this sense the words are used, not as conveying any meaning in themselves, but as an abbreviated form of expression for an idea which we cannot state completely without a good many words. But the expression is one with which we are so familiar that we are apt to forget this, and to think that the meaning must be contained in the expression itself. Take, for example, the case of a stone falling to the ground. At the instant it hits the ground we say it is moving at a certain definite rate. Let us for a moment suppose that the stone is stopped instantaneously, a condition unrealizable physically, but presenting a definite kinematic problem. Is the stone moving or not, at that instant? When I say that a body is moving at a certain instant, I mean (in accordance with the definition just

[1] *Encyclopaedia Britannica, sub* "Zeno."

quoted) that, if I compare its position at that instant, with its position at another instant sufficiently near to the first, I shall find that there has been a displacement. In the case considered, the stone is moving if the comparison be made with past time, it is not moving if the comparison be made with future time. But if the question be taken as complete in itself, it does not admit of an answer. The instant of stoppage is the boundary between the time during which there is motion, and that during which there is no motion; and this is true whether we suppose the stoppage to be abrupt or not.

It may be urged that a complete solution of the kinematic problem of the motion of a point is given by the curve of position, and that the tangent to the curve at any point gives a measure of the velocity at the corresponding instant. This is true, but this means of representation is only an analogy, and must not be pushed too far. Our idea of time is one dimensional; our idea of the velocity of a point is obtained by a combination of this with that of one-dimensional space, but the two concepts, time and space, in the original problem do not resemble one another in the same way as the two dimensions of space in the analogy. So far as the curve is merely at the point, it has no tangent. When I speak of the tangent at a point, I mean a line which meets the curve at the point—has both its co-ordinates the same as those of the curve—and touches the curve in the immediate neighbourhood. The line is "at the point" in a way which presents no analogy to "at the instant." So far as the analogy holds, the position of the point defines the part of the curve to be considered in approaching the limiting position of the chord, the instant defines the portion of time to be considered in determining the limit to which the velocity approaches.

To the phrase "motion at an instant," if it is regarded as connoting neither more nor less than is contained in the definition quoted above, I have no objection; some expression for this limit is necessary, and I cannot suggest a better; but, if it is maintained that the words can be understood without reference to this definition, I must confess that I for one am unable so to understand them.

S. A. SAUNDER.

A. 2. a.; L^1. 10. a.]

PORISMATIC EQUATIONS.

Read before the Mathematical Association on January 21st, 1899.

1. Some idea of a system of such equations may be given in order to clear the way.

In Professor Chrystal's *Algebra*, Chapter XIV., No. 18, p. 303, we have:

"Of the equations

$$\frac{x}{x^2-w^2}=\frac{y+z}{(m+1)w^2-(n+1)yz},$$

with two similar equations in y, z, where x, y, z are unequal; prove that any two imply the third." (Cayley.)

These equations are equivalent to two (and two only) symmetric relations involving all three variables,

$$\left.\begin{aligned} yz+zx+xy&=mw^2,\\ w^2(x+y+z)&=nxyz.\end{aligned}\right\}$$

Multiplying 1st by x,

$$(y+z)x^2+xyz=mw^2x;$$

arranging the 2nd,

$$w^2x+w^2(y+z)=nxyz$$

hence $$(x^2-w^2)(y+z)=(m+1)w^2x-(n+1)xyz.$$

2. Given n simultaneous equations between n unknown quantities, it may happen that it is impossible to solve these equations, owing to a certain peculiarity of their constitution which renders them either

(i.) INCONSISTENT; or

(ii.) INSUFFICIENT, should it be possible to bridge over the inconsistency.

In other words the given system may be transformed into another system of n simultaneous equations, in one of which the unknowns are absent altogether, causing it to become simply a relation between the coefficients of the given system.

3. The present paper will be restricted to the consideration of equations in *three* variables; that is to say, to systems of the form

$$F(y, z)=0, \quad F(z, x)=0, \quad \text{and} \quad F(x, y)=0,$$

where $F(y, z)\equiv F(z, y)$. These equations suggest themselves by some such geometrical problem as this: given a circle and a parabola, to describe a triangle inscribed to the circle and circumscribed to the parabola. If the circle does not pass through the focus of the parabola, the thing is impossible; if, on the other hand, the circle passes through the focus, there is an infinite number of solutions.

4. The late Professor Wolstenholme, in his *Collection of Problems*, pp. 61, 85, gives a note on this subject, in which are summarized (without proofs), the results of a paper read before

the London Mathematical Society many years ago. No new results are now to be added; but simple proofs will be given, and a certain point brought out and emphasized which is not noticed in the above paper.

5. Given the three equations:

$$\left.\begin{aligned}&\frac{a}{x_1x_2}+bx_1x_2+c+f(x_1+x_2)+g\left(\frac{1}{x_1}+\frac{1}{x_2}\right)+h\left(\frac{x_1}{x_2}+\frac{x_2}{x_1}\right)=0,\\&\frac{a}{x_1x_3}+\dots\dots\dots\dots\dots\dots\dots\dots\dots=0,\\&\frac{a}{x_2x_3}+\dots\dots\dots\dots\dots\dots\dots\dots\dots=0,\end{aligned}\right\}\dots\dots(\mathrm{A})$$

to find the relation between the coefficients a, b, c, f, g, h, it being supposed that x_1, x_2, x_3 are all unequal.

Multiplying up, we have

$$bx_1^2x_2^2+fx_1x_2(x_1+x_2)+cx_1x_2+h(x_1^2+x_2^2)+g(x_1+x_2)+a=0. \dots(\mathrm{i.})$$

Similarly,

$$bx_1^2x_3^2+fx_1x_3(x_1+x_3)+cx_1x_3+h(x_1^2+x_3^2)+g(x_1+x_3)+a=0.\dots(\mathrm{ii.})$$

Taking (ii.) from (i.), and dividing out by x_2-x_3 which is not zero,

$$bx_1^2(x_2+x_3)+f(x_1^2+x_1x_2+x_1x_3)+cx_1+h(x_2+x_3)+g=0. \dots(\mathrm{iii.})$$

Similarly,

$$bx_2^2(x_1+x_3)+f(x_2^2+x_2x_3+x_1x_2)+cx_2+h(x_1+x_3)+g=0. \dots(\mathrm{iv.})$$

Taking (iv.) from (iii.), and dividing by x_1-x_2 which is not zero,

$$b(x_1x_2+x_3x_1+x_2x_3)+f(x_1+x_2+x_3)+c-h=0, \dots\dots\dots(\mathrm{v.})$$

and this result being symmetrical the process cannot be carried further.

If we put

$$x_1+x_2+x_3=P,$$
$$x_2x_3+x_3x_1+x_1x_2=Q,$$
$$x_1x_2x_3=R,$$

we may write (v.)

$$bQ+fP+c-h=0, \dots\dots\dots\dots\dots\dots\dots\dots(\mathrm{v.})$$

Multiply (v.) by x_1 and take it from (iii.), then

$$-bx_1x_2x_3+h(x_1+x_2+x_3)+g=0, \dots\dots\dots\dots\dots(\mathrm{vi.})$$

Again multiply (v.) by x_1x_2 and take it from (i.),

$$-bx_1x_2x_3(x_1+x_2)-fx_1x_2x_3+h(x_1^2+x_1x_2+x_2^2)$$
$$+g(x_1+x_2)+a=0, \dots\dots(\mathrm{vii.})$$

Multiply (vi.) by x_1+x_2 and take (vii.) from it, then

$$h(x_1x_2+x_1x_3+x_2x_3)+fx_1x_2x_3-a=0,$$

or

$$hQ+fR-a=0, \qquad \text{(viii.)}$$

Comparing (v.), (vi.), (viii.),

$$\left.\begin{array}{l} fP+bQ\ldots+c-h=0, \\ hP\ldots-bR+g \quad\;\, =0, \\ \ldots+hQ+fR-a \;\; =0, \end{array}\right\} \qquad \text{(ix.)}$$

we have

$$h(c-h)-fg+ba=0,$$

or

$$h^2-ab-ch+fg=0, \qquad \text{(x.)}$$

The equations (ix.) are therefore inconsistent unless (x.) holds, and in that case they reduce to two relations between P, Q, R.

Note that $\quad aP+gQ+(c-h)R=0$:

also that the above relations (ix.) may be written

$$x+y+z+g/h=(b/h)xyz,$$

$$1/x+1/y+1/z+f/h=a/hxyz;$$

hence

$$(x+y+z+g/h)(1/x+1/y+1/z+f/h)=ab/h^2= \text{ constant.}$$

(See *Educational Times*, Question 13942.)

6. These results may be obtained in another way.

The three equations (A) necessitate the cubic in x

$$\frac{ax}{R}+b\frac{R}{x}+c+f(P-x)+g\left(\frac{Q}{R}-\frac{1}{x}\right)-\frac{h}{R}\{Px^2-(P^2-Q)x+R\}=0$$

being satisfied by the values $x=x_1$, $x=x_2$, $x=x_3$.

$$[Px_1^2-(P^2-Q)x_1+R$$
$$=x_1^2(x_1+x_2+x_3)-x_1(x_1^2+x_2^2+x_3^2+x_2x_3+x_3x_1+x_1x_2)+x_1x_2x_3$$
$$=-x_1(x_2^2+x_3^2).]$$

It must therefore be identical with $x^3-Px^2+Qx-R=0$. From a comparison we get the relations previously obtained.

7. It is proposed to take the system (A) as the *standard form* of a system of porismatic equations in three variables; and the solution {(ix.) and (x.)} as the *standard solution*. The following examples will illustrate the method, which it is unnecessary to go through in case after case.

8. Consider the equations given in Chrystal's *Algebra* Chapter XIV., p. 299 (bookwork);

$$p^2(y^2+yz+z^2)-pyz(y+z)+y^2z^2=0,$$

with similar equations in z, x and x, y.

Writing the above in the form

$$yz+p^2-p(y+z)+p^2(y/z+z/y)=0,$$

we have a comparison with the standard form

$$a=0,\ b=1,\ c=p^2,\ f=-p,\ g=o,\ h=p^2.$$

Hence $h^2-ab-ch+fg=p^4-p^4=0$, identically. Hence given any two of the above equations, the third can also be deduced.

The symmetric relations between x, y, z are

$$p(x+y+z)=yz+zx+xy,$$
$$p(yz+zx+xy)=xyz.$$

9. A little practice will enable one to discover very easily whether a set of algebraical equations are reducible to the standard form.

If they happen to be porismatic, as much attention should be given to the symmetric relations between the variables as to the relation between the constants.

The late Professor Wolstenholme in his *Problems*, Nos. 348-355, does not always note these relations; for instance, in No. 352,

$$yz+zx+xy=1/yz+1/zx+1/xy=a;$$

in 353 $$xyz=b=1/a,$$

in 355 $$1/yz+1/zx+1/xy+1/a^2=0.$$

10. Again denoting by the parameter t the point on the parabola $y^2=4mx$ whose co-ordinates are mt^2, $2mt$, it is known (or can be shown) that the equation of the circle circumscribing the triangle formed by the tangents at the points t_1, t_2, t_3 is

$$(x-m)(x-Qm)+y^2+my(R-P)=0,$$

where $$P=t_1+t_2+t_3,\ Q=t_2t_3+\dots+\dots,\ R=t_1t_2t_3.$$

Now if this triangle is inscribed to a fixed conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

the co-ordinates of any angular point $\{mt_1t_2,\ m(t_1+t_2)\}$ satisfy the equation to the curve.

Hence,

$$am^2t_1^2t_2^2+2hm^2t_1t_2(t_1+t_2)+bm^2(t_1+t_2)^2$$
$$+2gmt_1t_2+2fm(t_1+t_2)+c=0$$

with two other similar equations of the standard form.

The porismatic relation between the constants is

$$b^2m^2-4fhm+2bgm+ac=0,$$

and the relation between the variables

$$aQ=-(2hP+b+2g/m);\quad aR=bP+2f/m.$$

The equation to the circumscribing circle therefore involves a single variable parameter P in the first degree, and consequently passes through a second fixed point in addition to the focus of the parabola; as Mr. C. E. M'Vicker has shown in the October number of the *Gazette*. R. F. DAVIS.

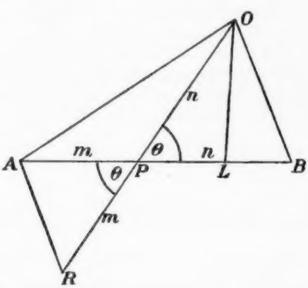
(*To be continued.*)

MATHEMATICAL NOTES.

71. [R. 4. d.] *Geometrical Proof of a Statical Theorem* (Minchin, *Statics*, p. 30, 1884).

Let OAB be a triangle, and P a point in the base AB such that

$$AP : PB = m : n.$$



Draw OL perpendicular to AB; and denote the angle OPB by θ.

Then $n(AL - PL) = m(PL + BL)$;

or $(m+n)PL = nAL - mLB.$

Dividing each side by OL,

$$(m+n)\cot\theta = n\cot OAB - m\cot OBA.$$

Draw AR parallel to OB meeting OP in R, so that $OP : PR = n : m$.

Then from the triangle AOR,

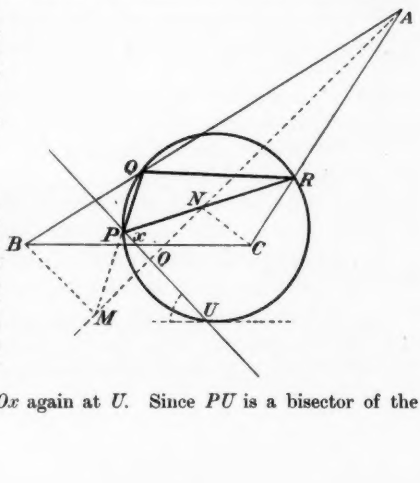
$$(m+n)\cot\theta = m\cot AOP - n\cot ARP$$
$$= m\cot AOP - n\cot BOP.$$

If $m = n$, the above formulae become

$$2\cot\theta = \cot OAB - \cot OBA$$
$$\text{or} \quad = \cot AOP - \cot BOP.$$

R. F. DAVIS.

72. [K. 2. E.] *A variable parallel is drawn to the base of a triangle ABC, meeting AB, AC in Q, R; BM, CN are perpendicular to the bisector of the vertical angle. If QM, RN meet at P, it is required to prove that the circumcircle of PQR touches the inscribed circle of ABC and also one of the exscribed circles.*



The triangles ABM, ACN are similar; also

$$AQ : QB = AR : RC,$$

therefore the figures $AQBM$, $ARCN$ are similar; so that QM, RN make equal angles with AO, and P lies on the line Ox bisecting MN at right angles. This line evidently bisects the base, and it is known to be the radical axis of the in-circle and the ex-circle opposite A.

Let the circle PQR meet Ox again at U. Since PU is a bisector of the

angle QPR, U is the midpoint of the arc QUR; thus the tangent at U is parallel to the base. This shows that *the circle PQR cuts the radical axis Ox at a constant angle.*

Again the squares of the tangents from M, N to the variable circle are numerically $MP.MQ$ and $NP.NR$ which, since $MP=NP$, have the ratio $MQ:NR$ or $AB:AC$. Hence the tangents from M, N to the varying circle are in a constant ratio. From this it is easy to deduce that *PQR is orthogonal to a fixed circle, with respect to which M, N are inverses.* Now M, N are known to be the common inverse points of the in-circle and ex-circle; hence PQR is orthogonal to some fixed circle of the coaxial system to which these circles belong.

But we have seen that PQR cuts the radical axis of this system at a constant angle; hence, finally, it must touch a pair of circles of the system.

This pair may easily be identified by taking a particular case. Suppose QR to coincide with the base of the triangle; then P is at infinity, and PQR opens out into the line BC. It is evident then that since every straight line touches two circles of the coaxial system, and two only, these circles must be the in-circle and ex-circle themselves.

(a) By supposing P to coincide with x, PQR becomes the Nine-point circle, which therefore touches the in- and ex-circles.

(b) By taking BM', CN' perpendicular to the other bisector of the vertical angle a theorem similar to the above may be obtained in which the enveloped circles are *both ex-circles* of ABC. C. E. M'Vicker.

EXAMINATION QUESTIONS AND PROBLEMS.

Our readers are earnestly asked to help in making this section of the Gazette *attractive by sending either original or selected problems.*

Solutions should be sent within three months of the date of publication. They should be written clearly on one side of the paper. Contractions not intended for printing should be avoided. Figures should be drawn with the greatest care on as small a scale as possible, and on a separate sheet.

The question need not be re-written, but the number should precede every solution.

The source of problems when not otherwise indicated is shown by —C. (Cambridge), O. (Oxford), D. (Dublin), W. (Woolwich), Sc. (Science and Art Department), etc.

300. P and Q are the centres of the squares inscribed in a right-angled triangle.

(α) Prove the projections of PQ on the three sides are concyclic.

(β) Given PQ and side of either inscribed square, construct triangle. W. S. Cooney.

301. Find (geometrically) the locus of the foci of conics touching two fixed straight lines in fixed points. C. V. Durell.

302. PL, PM, PN are perpendiculars on the sides of a triangle ABC from any point P on the bisector of A. MN meets BC in X. If D be mid-point of BC, show that $DL:DX=(b-c)^2:a^2$.

R. W. Genese.

303. Find the locus of the centre of a rectangular hyperbola touching three straight lines; applying the principle that a circle whose ordinates are stretched in the ratio $1:\iota$ becomes a rectangular hyperbola. W. J. JOHNSTON.

304. If a, b, c, d are integers, and b, d contain no square factors, under what conditions will $\sqrt{a\sqrt{b}+c\sqrt{d}}+\sqrt{a\sqrt{b}-c\sqrt{d}}$ be the fourth root of an integer? A. LODGE.

305. If two identically equal polygons $ABC\ldots A'B'C'\ldots$ are situated anywhere in the same plane, prove that either the perpendicular bisectors of AA', BB', $CC'\ldots$ are concurrent, or the mid-points of AA', BB', $CC'\ldots$ are collinear. What is the corresponding theorem for any two similar polygons situated in the same plane. F. S. MACAULAY.

306. A, B, and C have n articles to divide between them subject to the condition that B must not have more articles than either A or C. Show that the number of arrangements is given by the coefficient of x^n in the expansion of

$$\frac{1}{(1-x)^2(1-x^3)}.$$

If, however, the condition be that B is not to have fewer than either A or C, show that the generating function is

$$\frac{1+x^2}{(1-x)(1-x^2)(1-x^3)}.$$ P. A. MACMAHON.

307. The shortest side of a rational plane triangle having integral sides prime to each other cannot be less than 3.
ARTEMAS MARTIN.

308. An ellipse is inscribed to a convex quadrilateral $ABCD$. If α denote the ratio of the product of the focal distances of A to the product of the sides meeting at A, with similar notation for the remaining vertices, then $\alpha+\beta=\beta+\gamma=\gamma+\delta=1$.
C. E. M'VICKER.

309. Similar isosceles triangles are described on the sides of a triangle ABC. Show that the triangle formed by their vertices is in perspective with ABC, and find the locus of the centre of perspective, and the equation to the axis of perspective.
T. S. MUNSTER.

310. A and B are exploring a desert with the object of penetrating as far into the interior as possible. Each man carries provisions for 20 days' journey, find the greatest distance penetrated. C. PENDLEBURY.

311. The number *abcd* is such that $(ab+cd)^2=abcd$.
Ex. $(20+25)^2=2025$. What four-figure numbers possess this property? R. PENDLEBURY.

312. A triangle is inscribed in an ellipse and circumscribed to a confocal ellipse. Prove that the ratio of the radii of the in- and circum-circles of the triangle is constant. C. F. W. SANDBERG.

313. Through a given point in the plane of a triangle to draw three lines terminated by the sides of the triangle, two and two, such that their extremities shall lie on a circle. J. A. THIRD.

314. ABC is a triangle. AL any line meeting BC in L, on it AP is taken so that $AP.AL=k^2$. From any point D on AB, DE is drawn parallel to AL to meet BC in E, and DF is taken so that $DF.DE=k^2$. BF produced cuts AL in G. Through P, G any circle is drawn; prove that the tangent AT to this circle is equal to DF. R. TUCKER.

315. If the pedal lines of D, E, F with respect to a triangle ABC meet in a point P, then (1) these pedals are perpendicular to EF, FD, DE; (2) P bisects the orthocentre-join of ABC and DEF; (3) the pedal lines of A, B, C for the triangle DEF pass through P. C. E. YOUNGMAN.

316. If x, y, z, u are four positive quantities,

$$3\Sigma(y+z+u)^{-1} \not< 16(x+y+z+u)^{-1},$$

E. M. LANGLEY.

and the minimum value of

$$\Pi(x+1) \text{ is } \frac{256}{81} \text{ if } \Sigma\frac{u}{u+1}=1. \quad \text{(Queen's, 1898, C.)}$$

If a, b, c, d, e, are all positive, $\Sigma\left(\frac{abc}{de}\right)^4 > \Sigma ab^2c$.

(St. Cath's., 1896.)

317. If $\dfrac{e^{ax}-1}{e^a-1}=x+aP_2(x)+\dfrac{a^2}{\underline{|2}}P_3(x)+\dfrac{a^3}{\underline{|3}}P_4(x)+\dots$

then

$$-P_n(-x)=P_n(x)-(n-1)xP_{n-1}(x)+\frac{(n-1)(n-2)}{\underline{|2}}x^2P_{n-2}(x)+\dots$$
$$+(-1)^{n-2}(n-1)x^{n-2}P_2(x)+(-1)^{n-1}x^n.$$

(Trinity, C., 1896.)

318. From any point P on the outer of two similar coaxial ellipses, tangents PT, PT' are drawn to the inner, and a circle of

given radius is described with the centre P. Find the envelope of the four tangents from T, T' to the circle.

(Gonv. and Caius, 1894.)

319. (a) A shot of mass m is fired from a smooth bore gun whose carriage moves freely on a level plain; the inclination of the gun is fixed at a, the impulse exerted by the powder is I, and the mass of gun and carriage is M. Find the range on the plane, and determine the initial velocity of the shot.

(Trinity, C., 1897.)

(b) A spherical shell is dropping vertically, and just as it reaches the ground with a velocity V, it explodes so that each fragment of surface is driven out with a relative normal velocity u. Prove that if $u > V$ the fragments will be distributed on the ground within a circle of radius

$$\frac{1}{8g}[\sqrt{V^2+8u^2}-3V][8u^2-2V^2-2V\sqrt{V^2+8u^2}]^2$$

where the radius of the shell is neglected in comparison with this length. (Trinity, C., 1896.)

SOLUTIONS.

Solutions are wanted for Nos. 171, 252, 258, 265–8, 271, 274–9, 282, 283, 285, 287, 288, 291, 299.

249. [I. 9. c.]. *If the smallest prime factor of a number be not less than the cube root of that number, then the remaining factor is also prime.* E. Hill.

Solution by J. L. Thomas, R. F. Davis, J. C. Palmer, C. F. Sandberg.

Let N be the given number, a its smallest prime factor, $N=aP$. By hypothesis $a>N^{\frac{1}{3}}$, or $a^3>N$, that is $>aP$. Hence $a^2>P$; now if P be separable into two factors other than $1 \times P$, one of them must be less than a. This is contrary to the hypothesis and therefore P must be prime.

250. [L². 5. c.]. *If a circular cone be cut by a plane in an elliptic section, and if AB, AC are the greatest and least slant heights (from the vertex to the edge of the base), show that the volume of the cone so cut off is the G.M. of the right cones, cut off from the same cone, whose slant heights are respectively AB, AC.* A. Lodge.

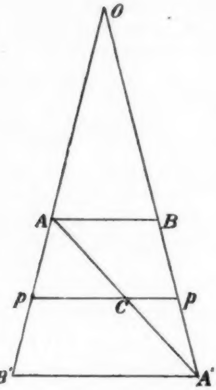
Solution by C. E. M'Vicker, J. C. Palmer.

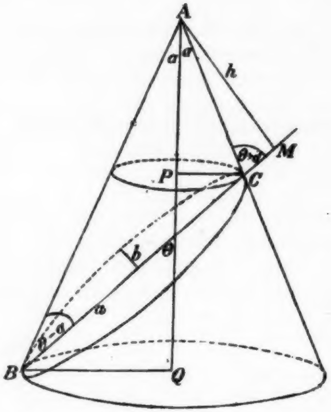
Suppose the plane of the paper to contain the axis of the cone, and axis major of oblique section (AA').

The middle point C of AA' is the centre of the elliptic base. Through A, C, A' draw planes perpendicular to the paper, cutting off right cones, their traces being AB, pp', $A'B'$.

The axis minor of (AA') is clearly the intersection of the planes AA', pp', and its semi-length is therefore a mean proportional to Cp, Cp'. But these

are the halves of AB, $A'B'$ respectively; hence axis minor of (AA') is the geometric mean of AB, $A'B'$.





$$\text{The triangle } OAB : OAA' = OB : OA',$$
$$= OA : OB',$$
$$= OAA' : OA'B';$$

therefore the triangle OAA' is the G.M. of OAB, $OA'B'$.

Again the volume of any cone cut off varies as the product of the axis minor of its base and triangular section by the paper; and as the minor axes and also the triangular sections have been shown to form continued proportions, proposition follows.

Solution by E. P. Barrett.

Draw APQ the axis of the cone cutting the base in Q and the circular section at C in P.

Then if BC and AQ intersect in θ, and $2a$ be vertical angle of cone $A\hat{B}C = \theta - a$. Draw $AM(=h)$ perpendicular to BC. Then $ACM = \theta + a$.

Then $$b^2 = CP \,.\, AQ = h^2 \sin^2 a / \sin \overline{a+\theta} \,.\, \sin \overline{\theta - a},$$

$$2a = BM - CM = h \sin 2a / \sin \overline{\theta + a} \,.\, \sin \overline{\theta - a};$$

$\therefore$ volume of cone with elliptic base,

$$V = \frac{\pi h}{3} \,.\, ab = \frac{\pi h^3 \sin^2 a \,.\, \cos a}{3 \,.\, (\sin \overline{\theta + a} \,.\, \sin \overline{\theta - a})^{\frac{3}{2}}}.$$

Product of volumes of right sections,

$$V_1 \,.\, V_2 = \frac{\pi^2}{9} \,.\, PC^2 \,.\, BQ^2 \,.\, AP \,.\, AQ$$

$$= \frac{\pi^2}{9} \,.\, \frac{h^4 \sin^4 a}{\sin^2 \overline{\theta + a} \,.\, \sin^2 \overline{\theta - a}} \,.\, \frac{h^2 \cos^2 a}{\sin \overline{\theta + a} \,.\, \sin \overline{\theta - a}}$$

$$= V^2.$$

Q.E.D.

251. [L^1. 11. c.; N. 1. e.]. *AA', BB' are diameters of a rectangular hyperbola, and P is any point on the curve within the parallelogram $ABA'B'$. Prove that $PA \,.\, PA' + PB \,.\, PB' = AB \,.\, AB'$. [As P passes along the curve to the outside, the sign of the term which vanishes* en route *must be changed.*]

C. E. M'VICKER.

Solution by W. J. JOHNSTON.

Let CP, CA, etc., be *complexes* the transverse axis being the initial line. Then

$$AP \,.\, A'P = (CP - CA)(CP + CA) = CP^2 - CA^2, \quad BP \,.\, B'P = CP^2 - CB^2;$$
$$\therefore \ AP \,.\, A'P - BP \,.\, B'P = CB^2 - CA^2 = AB \,.\, A'B.$$

Consider the case when P is on the same branch of the hyperbola as A and B and *not* between A and B. Then AP, $A'P$ are equally inclined to the asymptotes, so that the phase of $AP \,.\, A'P = \frac{\pi}{2}$. This is also the phase of $BP \,.\, B'P$ and $AB \,.\, A'B$. Therefore

$$\text{mod}\,(AP \,.\, A'P) - \text{mod}\,(BP \,.\, B'P) = \text{mod}\,(AB \,.\, A'B),$$

which is the theorem.

The method employed in this solution is that explained by Bellavitis in his treatises on equipollences.

The theorem is easily verified by co-ordinates. If the points P and A are $(a \sec\phi,\ a\tan\phi)$, $(a\sec\alpha,\ a\tan\alpha)$, the hyperbola being $x^2 - y^2 = a^2$, then

$$PA \,.\, PA' = 2a^2(\sec\phi\tan\phi - \sec\alpha\tan\alpha).$$

Solution by PROPOSER.

Since AA' is a diameter and P is on the curve, the asymptotes are parallel to the bisectors of angle APA', and for same reason parallel to bisectors of BPB'. These angles being bisected by the same straight lines, it easily follows that the opposite sides of the parallelogram subtend supplementary (fig. 1) or equal (fig. 2) angles at P.

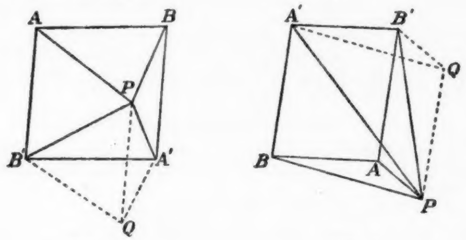


FIG. 1. FIG. 2.

Draw PQ equal and parallel to AB'. Join QA'. QB'.

By the above it can be seen that a circle passes through $PQA'B'$. Hence by Ptolemy's theorem,

$$QA' \,.\, PB' + QB' \,.\, PA' = PQ \,.\, A'B'.$$

Again, $QA' = PB$; $QB' = PA$; $PQ = AB'$; whence result.

253. [R. 7. a.]. *Find the hodographs of points on the rim or spokes of a wheel moving uniformly in a straight line, and of a point on the circumference of a circle rolling on a fixed circle.*

R. F. MUIRHEAD.

Solution by R. F. MUIRHEAD.

The hodograph of the centre of a circle which rolls uniformly on another is a circle described uniformly.

The relative motion of a point on the circle has a hodograph which is a circle described uniformly.

Compounding these for the hodograph required, we see it is an epicyclic curve, and therefore an epicycloid, the elements of which are easily calculated, as the angular velocity of rolling in it is constant.

From the same point of view, the hodograph of a point fixed on a circle rolling on a straight line, c from its centre, radius a, is a circle, radius $c\omega$, described uniformly, the origin of the hodograph being a point distant $a\omega$ from its centre.

254. *Walking from A to B, driving from B to C, and riding from C to A takes me* $15\frac{1}{2}$ *hours; driving from A to B, riding from B to C, and walking from C to A takes me* 12 *hours. Walking, the journey takes* 22 *hours, riding,* $8\frac{1}{4}$, *driving,* 11 *hours. To walk a mile, ride a mile, and drive a mile takes altogether* $\frac{1}{2}$ *hour. Find the rates at which I travel, and the distances A to B, etc.* (*Required a solution by Arithmetic.*) T. ROACH.

Solution by J. L. THOMAS and PROPOSER.

To walk the whole round, ride it, and drive it takes $41\frac{1}{4}$ hours.

In $\frac{1}{2}$ hour I can walk a mile, ride it, and drive it;

$\therefore$ the round is $82\frac{1}{2}$ miles; and I walk, ride, and drive at $3\frac{3}{4}$, 10, $7\frac{1}{2}$ miles per hour.

In $15\frac{1}{2}$ hours I walk all AB, $\frac{1}{2}$ BC, and $\frac{3}{8}$ CA; but all round takes 22 hours;

$\therefore$ in $6\frac{1}{2}$ hours I walk $\frac{1}{2}$ BC and $\frac{5}{8}$ CA. ....(1)

In 12 hours I walk $\frac{1}{2}$ AB, $\frac{3}{8}$ BC, and all CA; but half round takes 11 hours;

$\therefore$ in 1 hour I walk $\frac{1}{2}$ CA, less $\frac{1}{8}$ BC;

$\therefore$ in 4 hours I walk 2 CA, less $\frac{1}{2}$ BC. ....(2)

(1)+(2) gives in $10\frac{1}{2}$ hours I walk $2\frac{5}{8}$ CA;

$\therefore$ CA is 4 hours walk, or 15 miles;

$\therefore$ BC is 30 miles, and AB $37\frac{1}{2}$ miles.

255. [L^1. 6. b.]. *O is the centre of curvature of a point P on an ellipse, centre C; the tangent at P meets the axes produced in T, t; and CtQT is a rectangle. Show that* (1) *CO is perpendicular to PQ;* (2) *CO/PQ is the cotangent of the angle between CP and Tt.* E. P. ROUSE.

Solution by R. F. DAVIS.

Draw CU perpendicular to PQ, and produce it backwards through C to meet the normal at P in O. Draw CR and QL equal perpendiculars upon the tangent at P.

Then $PO : PU = PQ : QL = PQ : CR$. Therefore

$$PO = PU \,.\, PQ/CR = PT \,.\, Pt/CR = CD^2/CR = \text{radius of curvature at } P.$$

Again $CO : PR = PQ : QL = PQ : CR$. Hence $CO/PQ = PR/CR =$ cotangent of angle between CP and Tt.

256. [K. 4.]. *The base of a triangle is given in position and magnitude, and its vertex lies on a given straight line. If the sum of the sides is given, construct the triangle.* J. V. THOMAS.

Solution by R. F. DAVIS, J. C. PALMER, C. F. SANDBERG.

Let S, S' be the extremities of the given base, C its middle point. Upon SS' set off $CA = CA' =$ semi-sum of sides. Produce CA, CA' to X, X' respectively, so that $CS \,.\, CX = CS' \,.\, CX' = CA^2$. Let the perpendiculars to the base through C, X, X' meet the given straight line which is the locus of the vertex in Q, R, R' respectively. Then if P be a possible position of the vertex, and PN perpendicular to CA,

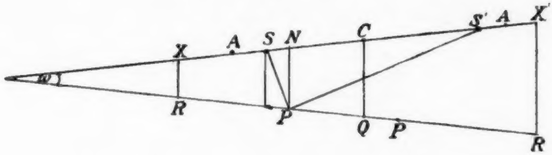
$$S'P^2 \backsim SP^2 = (S'P+SP)(S'P \backsim SP)$$
$$= 2CA(S'P \backsim SP)\,;$$

also $$= S'M^2 \backsim SN^2 = 4CN\,.\,CS.$$

Therefore $$S'P \backsim SP = 2\lambda\,.\,PQ\,;\ \text{if } \lambda = e\cos\omega,\ \text{where } e = CS/CA.$$

Again $$S'P + SP = 2CA = 2eCX = 2\lambda QR,$$

so that $$SP = \lambda PR \text{ and } S'P = \lambda PR'.$$



Hence a circle described upon the segment of SR, whose extremities divide SR internally and externally in the ratio $\lambda : 1$, will intersect the given straight line in points P, P', which are the vertices required.

257. [K. 10. b.]. *The points P, Q, R, S are taken on two orthogonal diameters of a circle, centre O, so that $OP = OQ = OR = OS$; AB, BC, CD, DA are the polars of the points. AP produced meets BR in p; prove that the locus of p and the analogous points is a cubical parabola, and that AP, BR and the analogous lines envelope hyperbolas.* R. TUCKER.

Solution by W. S. COONEY and PROPOSER.

Let r = radius of circle, $OP = m$ = etc. Perpendicular from O on $AB = a$, etc.

Equation of AP (join of $-a, a\,;\ o, m$), $(a-m)x + ay = am = r^2$,

" BR, $(a+m)x - ay = r^2$;

$$\therefore \text{ for } p,\ x = m,\ y = m^2/a, \text{ or } y = \frac{x^3}{am} = \frac{x^3}{r^2}\,;$$

$\therefore$ locus of p, etc., is as stated.

The equation to AP $\left(\text{writing } m = \dfrac{r^2}{a}\right)$ is

$$a^2(x+y) - ar^2 - r^2x = 0\,;$$

$\therefore$ envelope is $$r^4 + 4(x+y)r^2x = 0,$$

or $$4x(x+y) + r^2 = 0,$$

an hyperbola, asymptotes the axis of y and bisector of angle between the axes. Similarly envelope of BR is $4x(x-y) + r^2 = 0$.

259. [I. 9. c.]. *There is no number of six figures which does not contain factors prime to* 111 111 1. W. P. WORKMAN.

Solution by J. L. THOMAS.

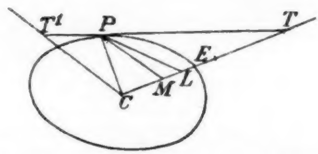
Any prime factor of 111 111 1 is a factor of 999 999 9;
$\therefore$ its reciprocal is a decimal recurring in 7 figures;
$\therefore$ the number, necessarily odd, is of the form $14n+1$.
And as 111 111 1 $= 239 \times 4649$, all numbers containing no factors prime to 111 111 1 are of form $(239)^a(4649)^b$.

For $a=0, b=1$; $a=0, b>1$; $a=1, b=0$; $a=1, b>0$; $a=2, b=0$; $a=2, b>0$; $a=3, b=0$, the result has respectively :—4; more than 6; 3; more than 6; 5; more than 6; 8 figures; $\therefore$ etc.

Note by PROPOSER.

The question may take the forms: There are only nine fractions of the form $1/N$ which have a 7-figure period; *or*, of the numbers 2 to 9 999 998 (inclusive) all but 8 contain factors prime to 9 999 999.

260. [L[1]. 3. b.; 4. c.]. *The tangent to an ellipse at P meets the equiconjugate diameters at T, T'. The tangent at P to the circle T'CP meets CT in L. Show that $PL^2=CL.TL$.* W. F. BEARD.



Solution by W. S. COONEY, H. G. MAYO, and PROPOSER.

Draw PM parallel to CT', and let CT meet curve in E.

$$\angle CPL=\angle CT'P=\angle MPT, \text{ and } \angle CPM=\angle LPT;$$

$$\therefore \frac{PC.PL}{PM.PT}=\frac{CL}{MT}, \text{ and } \frac{PC.PM}{PL.PT}=\frac{CM}{LT};$$

$$\therefore \frac{CL.LT}{CM.MT}=\frac{PL^2}{PM^2} \quad (a),$$

but PT being a tangent, $CM.CT=CE^2$;

$$\therefore CM.MT=CE^2-CM^2=PM^2,$$

for PM and CM are co-ordinates of P referred to the equiconjugate diameters;

$$\therefore PM^2+CM^2=CE^2;$$

$\therefore$ from (a), $$CL.LT=PL^2.$$

261. [L[1]. 1. c.]. *If opposite sides of a hexagon ABCDEF be parallel, and another hexagon PQRSTV be formed by intersections of AC and BD, BD and CE, CE and DF, etc., prove that Pascal's theorem holds true for all the hexagons (or hexagrams) formed by the points A, B, C, D, E, F, and that Brianchon's theorem holds for PQRSTV.* W. S. COONEY.

Solution by PROPOSER and H. G. MAYO.

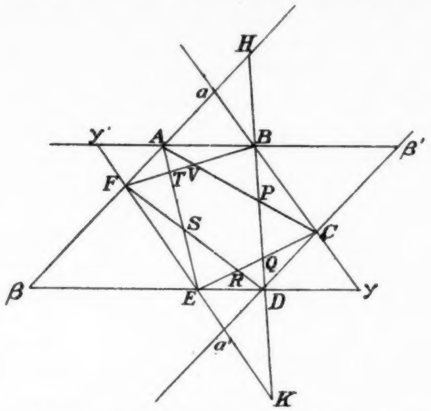
The parallel sides being produced, the $\triangle^s$ $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$ are evidently in perspective, the axis being the line at infinity; therefore all the conditions necessary for the application of Pascal's theorem to the six points $ABCDEF$ are fulfilled.

If we take the three hexagons $AEBDFC$, $CFBDAE$, and $BECADF$, the three Pascals are respectively PS, QT, and RV, but by Steiner's theorem, these hexagons being similarly circumstanced, the three Pascals are concurrent; therefore $PQRSTV$ is a Brianchon hexagon.

In this particular hexagon it can be proved independently that PS, QT, and RV are concurrent, for, anharmonic ratio $ATSE$=anh. $HBDK$

$$=\frac{DH}{HB}\times\frac{BK}{KD}=\frac{C\alpha}{\alpha B}\times\frac{C\alpha'}{\alpha' D}=\frac{F\alpha'.F\alpha}{\alpha A.\alpha' E}=\text{anh. } BPQD;$$

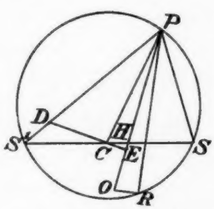
therefore pencil $D.ATSE$=pencil $E.BPQD$, and $B.ATSE=A.BPQD$. Now if PS and QT intersect in O, AQ and BS in X, AD and BE in Y,



TD and PE in Z, from quadrilateral $ABQS$, V, X, O are collinear, and from quadrilateral $TPDE$, R, Z, O are collinear; therefore, from the equianharmonic pencils, V, X, O, Y, Z, R are collinear, therefore, etc.

262. [L^1. 6. a.]. *P is a point on an ellipse, foci S, S'. PR, the symmedian through P of the triangle SPS', cuts the circumcircle in R. RO perpendicular to PR cuts the normal at P in O the centre of curvature at P.* R. F. Davis.

Solution by W. S. Cooney.



Let C be centre of ellipse, $PC=m$, $PR=d$, $PH=N$ (normal), $SP=f'$, $S'P=f$. Draw DCE perpendicular to OP meeting OP in E and $S'P$ in D. Since PR passes through symmedian point of $\triangle SPS'$, $\angle RPS=\angle CPS'$, $\therefore$ from similar $\triangle^s$ RPS and CPS', $\dfrac{d}{f'}=\dfrac{f}{m}$, and from similar $\triangle^s$ CPE and OPR, $\dfrac{OP}{d}=\dfrac{m}{PE}$.

$$\therefore\ \frac{OP}{f'}=\frac{f}{PE},\quad \therefore\ OP=\frac{ff'}{PE}=\frac{ff'}{\frac{1}{2}(f+f')\cos\frac{1}{2}SPS'};\ \text{since } PD=\tfrac{1}{2}(f+f'),$$

but HP or $N=\dfrac{ff'\cos\frac{1}{2}SPS'}{\frac{1}{2}(f+f')}$, $\therefore$ $OP=\dfrac{N}{\cos^2\frac{1}{2}SPS'}$ = radius of curvature.

263. [K. 8. a.; 11. a.]. *$ABCD$ is a quadrilateral with given sides, on a fixed base AB. A triangle is inscribed in a fixed circle, sides parallel to BC, CD, DA respectively. Show that each side of the triangle envelopes a circle, and that the four circles are coaxial.* A. C. Dixon.

Solution by C. E. M'Vicker.

Take axes parallel and perpendicular to AD as in figure.

Let PQR be the variable triangle, r the radius of the fixed circle concentric with the origin to which it is inscribed.

Take any point on the y-axis having an ordinate h; let ρ be its distance from QR, and let p be the perpendicular from origin on same line.

By projections, $p=\rho+h\cos D$,

and $p=r\cos QPR=-r\cos B$;

$\therefore\ h\cos D+\rho+r\cos B=0$, ...(1)

Again $a^2+b^2-2ab\cos B=AC^2$,

and $c^2+d^2-2cd\cos D=AC^2$;

$\therefore\ 2cd\,.\cos D+(a^2+b^2-c^2-d^2)-2ab\,.\cos B=0$. ....(2)

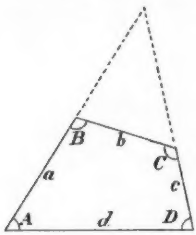


FIG. 1.

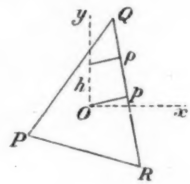


FIG. 2.

Now choose h so that $h/r=-cd/ab$, and we find by (1) (2) that

$$\frac{h}{2cd}=\frac{\rho}{a^2+b^2-c^2-d^2}=\frac{-r}{2ab}, \quad \text{..............................(3)}$$

Having thus found constant values for h, ρ, we see that QR envelopes the fixed circle $x^2+(y-h)^2=\rho^2$ in which h, ρ have the values given by (3). Similarly the other sides RP, PQ touch fixed circles.

The radical axis of the circle enveloped by QR and the circle $x^2+y^2=r^2$ is $2hy=h^2-\rho^2+r^2$, which by (3) may be written

$$8abcd\,.\,y/r=a^4+b^4+c^4+d^4-2\Sigma(b^2c^2+a^2d^2).$$

The symmetry of this equation proves rest of theorem.

264. [K. 2. a.]. *The circles in* 248 *cut on the Simson line of M.*

Solution by PROPOSER.

Let o, o' be centres of circles bcP, caQ, being respectively on MP, MQ. These circles cut in c.

The triangles oPc, $o'Qc$ are isosceles; the quadrilaterals $MPCc$, $MQCc$ are cyclic;

$$\therefore\ ocP=oPc=MCc=MQc=o'cQ;$$

$$\therefore\ oco'=PcQ \text{ whence etc.}$$

269. [J. 1. b, a]. *There are* 15 *boat clubs:* 2 *of the clubs have each* 3 *boats on the river,* 5 *others have* 2, *and the remaining* 8 *have* 1; *find an expression for the number of ways in which a list can be formed of the order of the* 24 *boats, observing that the second boat of a club cannot be above the first, or the third above the second.* R. F. MUIRHEAD.

Solution by H. P. KNAPTON.

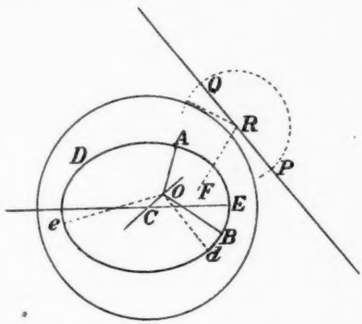
The boats belonging to a club do not undergo permutations among themselves.

$\therefore$ number required is $24!/[(3!)^2\,(2!)^5]$

270. [L^1. 5. d.]. *PQ are points of inverse co-ordinates* $\left(i.e.\ h, k, \frac{a^2}{h}, \frac{b^2}{k}\right)$ *with respect to the axes of an ellipse; F is the pole of the line PQ. Through F a straight line is drawn making with either axis an angle equal to that made by PQ but in the opposite direction. Prove that it bisects PQ.*

Hence, given any diameter of the ellipse find groups of four points on the circumference from each of which groups the four normals drawn shall meet somewhere on that diameter. (*See Problem* 225.) E. P. ROUSE.

SOLUTION BY PROPOSER.



The equation to PQ is

$$\frac{y-k}{x-h}=\frac{h(k^2-b^2)}{k(h^2-a^2)}=m, \text{ suppose}$$

i.e., $xh(k^2-b^2)-yk(h^2-a^2)$
$$=a^2k^2-b^2h^2.$$

If f, g are the co-ordinates of F, the pole of this line,

$$f=\frac{a^2h(k^2-b^2)}{a^2k^2-b^2h^2},$$

$$g=-\frac{b^2k(h^2-a^2)}{a^2k^2-b^2k^2};$$

and the equation to a line through F, equally inclined with PQ to the axis, is

$$\frac{y-g}{x-f}=-m=-\frac{h(k^2-b^2)}{k(h^2-a^2)}.$$

If this line bisects PQ it must pass through the two points

$$\frac{a^2}{h}, k, \text{ and } h, \frac{b^2}{k};$$

and on substitution it is found that the co-ordinates of both points satisfy the equation. Q.E.D.

COR. Given any line CO through the centre, to draw normals which shall meet on CO.

Take *any* line PQ at right angles to CO, find its pole F, and through F draw a line equally inclined with PQ to meet PQ in R. If R is not outside the director-circle, take another line PQ.

If R is outside the director-circle describe a circle with centre R and radius equal to the tangent from R to that director-circle, cutting PQ in two points P and Q.

Tangents to the ellipse from one of these will give us two points A, B whose normals meet at some point on CO; call it O.

Tangents from the other, if both are outside the ellipse, will give two points D, E, such that, if d and e are taken diametrically opposite to them, normals from d and e will also meet at the same point O. For proof compare Problem 225.

272. [L^1. 1. c.]. *Show that No.* 214 *and the generalization are particular cases of Pascal's Theorem.* J. A. THIRD.

Solution by PROPOSER.

The Theorem and Mr. Blaikie's generalization are particular cases of Pascal's Theorem. $ABQLRC$ (the notation of the generalization being adopted) is a hexagon whose alternate vertices lie on two straight lines.

Hence the pairs of opposite sides AB and LR, BQ and RC, QL and CA intersect collinearly. But the first and third pairs intersect on the line at infinity. Consequently this is also true of the second pair.

REVIEWS.

Introduction to the Theory of Analytic Functions. By PROFESSORS J. HARKNESS, M.A., and F. MORLEY, Sc.D. (Macmillan and Co, 8vo., pp. xv. +366.) The present book, as the preface tells us, is not an abridged and elementary version of the *Treatise on the Theory of Functions* by the same authors, but an independent work. "It has been composed with different ends in mind, and deals in many places with distinct orders of ideas." As an introductory book on the Theory of Functions it is sure to be welcomed by a large number of readers. The difficulties of the subject are such that it requires long and patient study in order to obtain any degree of manipulative skill with its methods of operation. Hence it is very important that its study should be commenced as early as possible. We have carefully read the book in the hope of finding it suitable for school use; but we regret to say that our hopes have been disappointed. For school purposes the book contains much more than is wanted, a minor drawback it is true, but one not easily remedied. A much more serious objection is that in many places the reasoning or explanation is obscure, a result which appears to be solely due to the book having been too hastily produced. We give some instances, not to show that our objection is well founded, but to suggest to the authors the necessity of revision in a future edition.

Instead of formal definitions of terms in separate paragraphs we find only haphazard explanations which are difficult to find, even with the aid of the index. The last paragraph of § 1 reads, "Objects considered as a succession from left to right are *in positive order*; when considered from right to left *in negative order*." As we make no pretence of understanding this, we only suggest with diffidence that the following may be intended, *viz.*, having fixed on an order of succession for a given finite or infinite series of objects, and calling it the *positive order*, the reverse order may be called the *negative order*. If this is meant, the statement in the book is very misleading, and its sole effect is to puzzle an intelligent reader. In § 35 we read, "If we consider a moving point x as depending on the time t, then the derivate of x as to t, which we denote by $\dot{x}$, is the *velocity* of x at the time t." We have not been able to find out whether this is the enunciation of a theorem or a definition of velocity, whether t is a complex or a real variable, whether x is an actually moving point as is stated, in which case the context seems to involve contradictions, or whether the motion spoken of is that of the real point which represents x in the x-plane, in which case we do not see the need of defining the velocity. In § 109 it is implied that the infinite product $\Pi(1-a_n)$ may be divergent when a_1, a_2, ... are real positive quantities less than unity without any statement to the effect that when the product tends to zero it is said to be divergent. In § 107 we are told that a sheet of paper without holes is *simply connected*, and that one with holes is *multiply connected*. With this slender information Cauchy's theorem (§ 118) is enunciated for a *simply connected region*. The introduction of this phrase makes the enunciation not only obscure, but ineffective, and the proof unintelligible.

Chapter I., on the Ordinal Number System, is interesting and important; but we cannot but consider it as misplaced at the beginning of the book, and as thus forming a serious stumbling-block for learners. In the ordinal number system all real numbers, rational and irrational, are regarded as symbols for denoting order only, not as symbols for expressing how many or how much. It is shown that the decimal notation is excellently adapted for the purpose, and that the operations of ordinal addition and subtraction can be carried out without attaching any notion of magnitude to the numbers employed. We think that on this subject an error is likely to creep in which it is necessary to guard against. Order, or succession, cannot in itself be conceived without the notion of magnitude. To know the order of an aggregate of objects means that of any two of the objects we know

which one comes after the other; and this being so, there must be some answer to the question, how much after; otherwise order is rendered inconceivable and non-existent. We might, however, choose not to concern ourselves with this question; and, even if it were considered, we should, in the first instance, refer it directly to the objects, not to the marks or numbers which denote the order of the objects. Magnitude is not a necessary attribute of number; and, in the ordinal number system, the consideration of intrinsic magnitude is deliberately rejected, for the special purpose of examining the fundamental properties of numbers which are independent of magnitude.

From the titles of twenty-two chapters we can only make a small selection at random. Chapters III. and V. are on Bilinear Transformation; Chapter VI., Limits and Continuity; Chapter VIII., Convergence of Infinite Series; Chapter XII., Continuation of Power Series; Chapter XVI., Integration; and Chapter XX., Simple Algebraic Functions on Riemann Surfaces. In the bilinear transformation of a plane into itself the spirals discussed on p. 213 of the *Gazette* are shown to play an interesting part, and are accurately drawn in a figure. The diagrams throughout the book are excellent; and the index seems adequate.

We urge on the consideration of the authors the desirability of still further supplementing their work on the Theory of Functions by the issue of a Primer. They would thus make it possible to introduce the subject into schools, and would make it easy for the student to gain, at an earlier period, a wider horizon of mathematical thought. F. S. M.

An Elementary Treatise on Analytical Geometry. By W. J. JOHNSTON, M.A. (Crown 8vo, cloth, pp. xiv. + 425, reduced to 6s.) Clarendon Press, 1899. Mr. Johnston's excellent handbook for beginners was well worth re-issue. The usual methods are generally followed, but fuller illustration is found wherever the experience of the author has shown him the pitfalls that beset the ordinary student. Attention may be drawn to the method of projecting metrical properties (p. 424), and to the method of tracing a parabola from its equation, p. 272. The following errata may be useful. Page 20, Ex. 3—For $1\frac{23}{25}$ read $9\frac{3}{5}$. Page 35, line 1—For $\triangle OBP + \triangle OPA$ read $2\triangle OBP + 2\triangle OPA$. Page 104 last line la should be lu. Page 215, Ex. 5—For "tangent," read "normal." Mr. Johnston has been eminently successful in realizing his aim—an easy and gradual development of the principles of the subject. The type and general "get-up" are worthy of the Clarendon Press.

Modern Geometry of the Point, Straight Line, and Circle. An Elementary Treatise, by J. A. THIRD, M.A. (Blackwood, 1898.) In this book of 227 pages will be found all the elementary standard theorems of the modern geometry of the straight line and circle, and, as there is a fair index, it would be useful as a book of reference; but we hardly know for what class of students it is intended. The simpler parts of the work are to be found in almost all modern editions of the Elements of Euclid. The serious student should acquire some knowledge of Conics before attacking the more advanced portions of the treatise, and then he would be prepared to apply modern methods to the most important branches of the subject—projection and reciprocation—which are necessarily not included in Mr. Third's book. As a rule the explanations are clear, and the steps in the proofs of theorems short, and thus suited to young beginners, but there is great variety in this respect, and the departure from clear short steps often takes place where it is least desirable. As an illustration of this we note that on p. 9 a conclusion is come to in one step, and identically the same conclusion has three steps given to it on p. 123.

Mr. Third, recognizing the importance of students drawing their own figures, leaves several theorems without any figures. This principle, however, may be carried to excess. It hardly seems to be wise to refer the student on p. 6 to a figure on p. 180; and it may be merely annoying to have six closely printed pages of important theorems on the circles connected with the triangle, all referring to one composite figure in which not one circle is drawn, but it is quite inexcusable to have the same figure to do duty as a complete quadrilateral and also as a complete quadrangle.

The author, in his preface, acknowledges his indebtedness to Townsend and others, and states, what is quite reasonable, that "it is unnecessary to indicate the numerous

original sources of information drawn upon, as the material used may now be regarded as common property." But the latter principle ought not to be regarded as a justification for reproducing a long and complex theorem from a *recently* published work. Messrs. Richardson and Ramsey will find that Article 40 (with the exception of the first three lines, the references, and three different letters in the figure), is line for line for fourteen lines identical with theorem 30, Chap. II., of their treatise published in 1894.

The New Science and Art of Arithmetic. By A. Sonnenschein and H. A. Nesbitt, M.A. (Pp. x.—500, 4s. 6d.) Sonnenschein. The work of 1870 modified and brought up to date. Most teachers will find it well worth while to read a treatise so carefully thought out as this has been. The whole treatment of decimals and approximations (pp. 289-424) is excellent. The authors seem surprised that they have found no imitators in connecting L.C.M. with Euc. V.

BOOKS, MAGAZINES, ETC., RECEIVED.

Partitions of Numbers whose Graphs possess Symmetry. By Major MacMahon, F.R.S. ("Cam. Phil. Trans.," XVII., part ii., pp. 149-170.)

Memoir on the Theory of Partition of Numbers. Part II. By Major MacMahon, F.R.S. ("Phil. Trans. Roy. Soc.," vol. 192, pp. 351-401.)

Point Groups in a Plane, and their effect in determining Algebraic Curves. By F. S. Macaulay, D.Sc. ("Proc. L.M.S.," vol. xxix., Nos. 654, 5.)

On the Intersections of Plane Curves. By F. S. Macaulay, D.Sc. ("Bull. Amer. Math. Soc." Second series. Vol. iv., No. 10, pp. 540-4.)

London Matriculation Directory. No. XXV. Jan. 1899. ("University Correspondence College." 1s.)

Analytical Geometry. By W. J. Johnston. (Clarendon Press, 1893. Reissue at 6s. pp. xiv. + 425.)

New Science and Art of Arithmetic. By Sonnenschein and Nesbitt. (Swan Sonnenschein & Co. 4s. 6d. pp. 500.)

About Fifth Power Numbers whose sum is a Fifth Power. By Artemas Martin, LL.D. ("Mathematical Magazine," vol. ii., No. 11, pp. 201-208.)

Sophus Lie. By Prof. G. B. Halsted. ("Science," vol. ix., pp. 447-8.)

The School World. No. 1. (Macmillan.) (*The Teaching of Algebra*, by Prof. G. B. Mathews, F.R.S.) 6d.

The Educational Review. New Series. Vol. I., No. 1. 4d. (203 Strand, W.C.)

Modern Geometry of the Point, Straight Line, and Circle. By J. A. Third, M.A. (Blackwood.) 3s.

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Periodico di Matematica, Dec. 1898—April 1899, and *Supplemento al Periodico di. M.* Fasc. IV.-VI. Jan.—April, 1899. Edited by Prof. Lazzeri. (Livorno.)

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